A theoretical calculation of the charge radius of He^3 was made recently by Pappademos[14] which is in reasonable agreement with the experimental rms values found in our studies. Some theoretical work on the electrodisintegration of He^3 has been carried out by Haybron[15] and this will be compared subsequently with the experimental inelastic continua now being studied at Stanford.

[14] J. M. Pappademos, Nucl. Phys. (to be published).
[15] R. M. Haybron (private communication).

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Variational Principles for Electromagnetic Potential Scattering*

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Variational methods are considered for the solution of the vector wave equation describing the field due to an arbitrary source placed in the neighborhood of an inhomogeneous absorbing medium. Variational principles for the tensor Green's function satisfying the point source equation, $\nabla\times\nabla\times\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}')-k^2\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}')+U(\mathbf{r})\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}')=-\mathbf{I}\delta(\mathbf{r}-\mathbf{r}')$, have been obtained in linear and exponential forms, analogous to the Altshuler principles for the scalar wave function. Stationary forms for the wave function and the scattering amplitude in the standard scattering problem (incident plane wave, outgoing solutions) are recovered when the point source recedes to infinity. For the special case of a spherically symmetric scatterer, the analysis leads to variational principles for the two independent lth-order phase shifts. The method is illustrated by a calculation of the fields internal to an axially symmetric potential.

I. INTRODUCTION

NEW methods of obtaining approximate solutions to the vector wave equation, based on variational techniques, are presented. The development of variational principles for electromagnetic scattering is, of course, not new, but most of the earlier methods have applied only to surface scattering[1–5]; i.e., the scatterers have been assumed to be perfect conductors. In spite of the considerable interest in scattering by dielectric obstacles, there exist relatively few variational principles applicable directly to the vector-potential-scattering problem. Those that do apply, notably the stationary forms based on the "reaction concept" of Rumsey,[6,7] may be useful in the calculation of cross sections but provide little information on the fields themselves at an arbitrary space point.

The objective of this paper is to present variational principles for the vector-scattering problem which are formal analogs of principles which have been found to be particularly useful in the scalar-scattering theory. In Sec. II, variational principles based on both the differential and integral equations for a generalized tensor Green's function are discussed, and in Sec. III variational principles for the field and dyadic-scattering amplitude are developed. The special case of a spherically symmetric scatterer is considered in Sec. IV, with the analysis leading to amplitude-independent variational principles for the two independent phase shifts required in the vector-scattering problem. An application of the formalism to the scattering of a plane wave by a complex axially symmetric potential is given in Sec. V.

As a slight notational simplification, the following convention is adopted throughout the paper. Unless otherwise specifically indicated, the product of a dyadic

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[1] The literature on the theory and applications of the variational method to problems of electromagnetic surface scattering is quite extensive, and no attempt will be made to provide an exhaustive list of references. Numerous additional references may be found in those cited in this article.

[2] H. Levine and J. Schwinger, Comm. Pure Appl. Math. **3**, 355 (1950).

[3] R. Kieburtz, A. Ishimaru, and G. Held, University of Washington, Department of Electrical Engineering, Tech. Rept. No. 45, 1960 (unpublished).

[4] R. F. Harrington, *Time-Harmonic Electromagnetic Fields* (McGraw-Hill Book Company, Inc., New York, 1961), Chap. 7.

[5] J. R. Mentzer, *Scattering and Diffraction of Radio Waves* (Pergamon Press, Inc., New York, 1955).

[6] V. H. Rumsey, Phys. Rev. **94**, 1483 (1954).

[7] M. H. Cohen, IRE Trans. Antennas Propagation **AP-9**, 193 (1955).

with a vector or another dyadic is understood to be the scalar product, and the dot customarily symbolizing scalar multiplication is dropped.

II. POINT-SOURCE VARIATIONAL PRINCIPLE

The differential equation which relates the electric field at any point of space to its sources is, for harmonic time dependence

$$\nabla\times\nabla\times\mathbf{E}-k^2\mathbf{E}+U(\mathbf{r})\mathbf{E}=-\mathbf{j}(\mathbf{r}), \tag{1}$$

where $k^2=\omega^2/c^2$, and where the scattering potential $U(\mathbf{r})$ is defined in terms of the dielectric constant of a bounded medium by $U(\mathbf{r})=k^2[1-\epsilon(\mathbf{r})]$; $\mathbf{j}(\mathbf{r})$ is proportional to the current density distribution. The solution of Eq. (1), subject to the outgoing-wave boundary condition, may be related to a tensor Green's function $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$ by

$$\mathbf{E}(\mathbf{r})=\int\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')\mathbf{j}(\mathbf{r}')d\mathbf{r}', \tag{2}$$

where $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$ is the dyadic field at the position $\mathbf{r}$ due to a unit dyadic point source at $\mathbf{r}=\mathbf{r}'$, and satisfies the equation

$$\nabla\times\nabla\times\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')-k^2\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')+U(\mathbf{r})\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')=-\mathbf{I}\delta(\mathbf{r}-\mathbf{r}'). \tag{3}$$

The field $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$ may also be represented by the integral equation

$$\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')=\mathbf{G}(\mathbf{r},\mathbf{r}')+\int\mathbf{G}(\mathbf{r},\mathbf{r}'')\mathbf{\Gamma}(\mathbf{r}'',\mathbf{r}')U(\mathbf{r}'')d\mathbf{r}'', \tag{4}$$

where $\mathbf{G}(\mathbf{r},\mathbf{r}')$ is the usual free-space dyadic Green's function—i.e., the solution of (3) with $U(\mathbf{r})$ set equal to zero. It is shown in Appendix I that $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$ satisfies the symmetry condition

$$\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')=\mathbf{\Gamma}^T(\mathbf{r}',\mathbf{r}), \tag{5}$$

where $\mathbf{\Gamma}^T$ is the transpose of $\mathbf{\Gamma}$.

It is apparent from Eq. (2) that a stationary expression for $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$ provides a field $\mathbf{E}(\mathbf{r})$ having the property of stationarity; it is, therefore, sufficient to consider only variational principles for the Green's function $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$. The following expression defines a variational principle for the point-source problem[8]:

$$\mathbf{S}_1(\mathbf{r},\mathbf{p})=2\mathbf{\Gamma}(\mathbf{r},\mathbf{p})+\int\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')[\nabla'\times\nabla'\times-k^2+U(\mathbf{r}')]\mathbf{\Gamma}(\mathbf{r}',\mathbf{p})d\mathbf{r}'. \tag{6}$$

Note that $\mathbf{S}_1$ has the property that it reduces to $\mathbf{\Gamma}(\mathbf{r},\mathbf{p})$ for a choice of trial function equal to the exact $\mathbf{\Gamma}(\mathbf{r},\mathbf{p})$. The stationarity property may be established as follows: Consider an arbitrary variation $\delta\mathbf{\Gamma}$ from the true solution, so that, to first order in $\delta\mathbf{\Gamma}$,

$$\begin{aligned}\delta\mathbf{S}_1(\mathbf{r},\mathbf{p})&=2\delta\mathbf{\Gamma}(\mathbf{r},\mathbf{p})+\int\delta\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')[\nabla'\times\nabla'\times-k^2+U(\mathbf{r}')]\mathbf{\Gamma}(\mathbf{r}',\mathbf{p})d\mathbf{r}'+\int\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')\\&\quad\cdot[\nabla'\times\nabla'\times-k^2+U(\mathbf{r}')]\delta\mathbf{\Gamma}(\mathbf{r}',\mathbf{p})d\mathbf{r}'\\&=\delta\mathbf{\Gamma}(\mathbf{r},\mathbf{p})+\int\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')[\nabla'\times\nabla'\times-k^2+U(\mathbf{r}')]\delta\mathbf{\Gamma}(\mathbf{r}',\mathbf{p})d\mathbf{r}'.\end{aligned}$$

But from the transpose of Eq. (3),

$$\begin{aligned}[k^2-U(\mathbf{r}')]\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')&=[\nabla'\times\nabla'\times\mathbf{\Gamma}(\mathbf{r}',\mathbf{r})]^T+\mathbf{I}\delta(\mathbf{r}-\mathbf{r}')\\&=[\nabla'\times\nabla'\times\mathbf{\Gamma}^T(\mathbf{r},\mathbf{r}')]^T+\mathbf{I}\delta(\mathbf{r}-\mathbf{r}'),\end{aligned}$$

where the superscript T denotes the transpose, so that

$$\delta\mathbf{S}_1(\mathbf{r},\mathbf{p})=\int\{\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')\nabla'\times\nabla'\times\delta\mathbf{\Gamma}(\mathbf{r}',\mathbf{p})-[\nabla'\times\nabla'\times\mathbf{\Gamma}^T(\mathbf{r},\mathbf{r}')]^T\delta\mathbf{\Gamma}(\mathbf{r}',\mathbf{p})\}d\mathbf{r}'.$$

Use of the identity[9]

$$\begin{aligned}\{\mathbf{A}\cdot\nabla\times\nabla\times\mathbf{B}-[\nabla\times\nabla\times\mathbf{A}^T]^T\cdot\mathbf{B}\}_{ik}&=\partial_l[A_{il}\partial_jB_{jk}-A_{ij}\partial_lB_{jk}+(\partial_lA_{ij})B_{jk}-(\partial_jA_{ij})B_{lk}],\\&\qquad i,\ j,\ k,\ l=1,\ 2,\ 3.\end{aligned} \tag{7}$$

reduces the i, k component of the variation to a surface integral

$$\begin{aligned}\delta S_1(\mathbf{r},\mathbf{p})_{ik}&=\int dS_l'[\Gamma_{il}(\mathbf{r},\mathbf{r}')\partial_j\delta\Gamma_{jk}(\mathbf{r}',\mathbf{p})-\Gamma_{ij}(\mathbf{r},\mathbf{r}')\partial_l\delta\Gamma_{jk}(\mathbf{r}',\mathbf{p})\\&\quad+\partial_l\Gamma_{ij}(\mathbf{r},\mathbf{r}')\delta\Gamma_{jk}(\mathbf{r}',\mathbf{p})-\partial_j\Gamma_{ij}(\mathbf{r},\mathbf{r}')\delta\Gamma_{lk}(\mathbf{r}',\mathbf{p})],\end{aligned}$$

for which the surface of integration shall be chosen to be a sphere of large radius r'. All derivatives appearing in the integrand are to be taken with respect to the r' coordinates; i.e., $\partial_j\equiv\partial/\partial x_j'$.

In order to evaluate the surface integral, the asymptotic forms of $\mathbf{\Gamma}(\mathbf{r}',\mathbf{p})$, $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$ are needed. These are most conveniently obtained by using the explicit form of the free-space Green's function[2]

$$\mathbf{G}(\mathbf{r},\mathbf{r}')=(\mathbf{I}-k^{-2}\nabla\nabla')g(\mathbf{r},\mathbf{r}'), \tag{8}$$

where $\mathbf{I}$ is the unit dyadic and $g(\mathbf{r},\mathbf{r}')$ is the free-space scalar Green's function $e^{ik|\mathbf{r}-\mathbf{r}'|}/(-4\pi)|\mathbf{r}-\mathbf{r}'|$. Inserting (8) into the integral equation for $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$ and taking the limit as $r'\to\infty$, we find

$$\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')\underset{r'\to\infty}{\longrightarrow}\frac{e^{ikr'}}{-4\pi r'}\mathbf{A}(-\mathbf{k}'|\mathbf{r}),$$

[8] The operator, $[\nabla\times\nabla\times-k^2+U(\mathbf{r})]$ has been improperly written as a mixed vector-scalar quantity. Strictly speaking, this operator should be written in the somewhat lengthier dyadic form $[\nabla\nabla-\mathbf{I}\nabla^2-k^2\mathbf{I}+U(\mathbf{r})\mathbf{I}]\cdot$.

[9] The convention of an implied summation on repeated indices is used throughout.

where $\hat{k}'=\hat{r}'$, and

$$\mathbf{A}(-\mathbf{k}'|\mathbf{r})\equiv e^{-i\mathbf{k}'\cdot\mathbf{r}}\mathbf{I}_{\mathbf{k}'}+\int \mathbf{\Gamma}(\mathbf{r},\mathbf{r}'')\mathbf{I}_{\mathbf{k}'}U(\mathbf{r}'')e^{-i\mathbf{k}'\cdot\mathbf{r}''}d\mathbf{r}'' \quad (9)$$

is the solution to the standard scattering problem, representing a plane wave incident in the direction $-\mathbf{k}'$, scattered by the potential U and observed at the position $\mathbf{r}$. The symbol $\mathbf{I}_{\mathbf{k}'}$ represents the modified unit dyadic $\mathbf{I}-\hat{k}'\hat{k}'$. Similarly,

$$\mathbf{\Gamma}(\mathbf{r},\mathbf{p})\xrightarrow[r\to\infty]{}\frac{e^{ikr}}{-4\pi r}\mathbf{A}^T(-\mathbf{k}|\mathbf{p}).$$

Use of these asymptotic forms and the relation $dS_l'\partial_l=dS'\partial/\partial r'$ reduces the surface integral, after some cancellation, to

$$\delta S_1(\mathbf{r},\mathbf{p})_{ik}$$
$$=\int\frac{e^{ikr'}}{-4\pi r'}\Bigg\{A_{il}(-\mathbf{k}'|\mathbf{r})\bigg[\partial_j\bigg(\frac{e^{ikr'}}{-4\pi r'}\bigg)\delta\tilde{A}_{jk}(-\mathbf{k}'|\mathbf{p})$$
$$+\frac{e^{ikr'}}{-4\pi r'}\partial_j\delta\tilde{A}_{jk}(-\mathbf{k}'|\mathbf{p})\bigg]-\bigg[\partial_j\bigg(\frac{e^{ikr'}}{-4\pi r'}\bigg)A_{ij}(-\mathbf{k}'|\mathbf{r})$$
$$+\frac{e^{ikr'}}{-4\pi r'}\partial_jA_{ij}(-\mathbf{k}'|\mathbf{r})\bigg]\delta\tilde{A}_{lk}(-\mathbf{k}'|\mathbf{p})\Bigg\}dS_l'.$$

From Eq. (9) it is evident that $\mathbf{A}(-\mathbf{k}'|\mathbf{r})$ is transverse (in its right-hand indices) to the direction $\hat{k}'$, i.e., $\mathbf{A}(-\mathbf{k}'|\mathbf{r})\cdot\hat{k}'=0$. If the same transversality property is assumed for the trial functions, then, since $d\mathbf{S}'=dS'\hat{k}'$, each term of the integrand of the surface vanishes and the stationarity of $\mathbf{S}_1(\mathbf{r},\mathbf{p})$

$$\delta S_1(\mathbf{r},\mathbf{p})_{ik}=0,\quad i,k=1,2,3,$$

is proved.

In addition to the scalar analog of $\mathbf{S}_1$, Altshuler[10] has proposed an exponential form of variational principle for the point-source scalar wave function, a principle whose dyadic generalization is

$$\mathbf{S}_2(\mathbf{r},\mathbf{p})=\mathbf{\Gamma}(\mathbf{r},\mathbf{p})\exp\bigg\{\mathbf{I}+\mathbf{\Gamma}^{-1}(\mathbf{r},\mathbf{p})\int\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')\cdot[\nabla'\times\nabla'\times-k^2+U(\mathbf{r}')]\mathbf{\Gamma}(\mathbf{r}',\mathbf{p})d\mathbf{r}'\bigg\},\quad (10)$$

where $\mathbf{\Gamma}^{-1}(\mathbf{r},\mathbf{p})$ is the matrix inverse of the trial function $\mathbf{\Gamma}(\mathbf{r},\mathbf{p})$. The stationarity of $\mathbf{S}_2(\mathbf{r},\mathbf{p})$,

$$\delta\mathbf{S}_2(\mathbf{r},\mathbf{p})=0,$$

is most easily proved by recognizing that the exponent may be written as $\mathbf{\Gamma}^{-1}(\mathbf{r},\mathbf{p})\mathbf{S}_1(\mathbf{r},\mathbf{p})-\mathbf{I}$, and using the relation $\delta\mathbf{\Gamma}^{-1}=-\mathbf{\Gamma}^{-1}\delta\mathbf{\Gamma}\mathbf{\Gamma}^{-1}$. The dyadic exponential in Eq. (10) is, of course, defined by its infinite series representation

$$\exp\{\mathbf{D}\}=\mathbf{I}+\sum_1^\infty\mathbf{D}^n/n!.$$

Both $\mathbf{S}_1$ and $\mathbf{S}_2$ are based on the differential operator $(\nabla\times\nabla\times-k^2+U)$ and the trial function must, therefore, be meaningful throughout all space. However, if, in the variational expressions, the point-source wave function $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$ is replaced by its integral equation, two stationary expressions are obtained in which the differential operator no longer appears. These alternative forms of $\mathbf{S}_1$ and $\mathbf{S}_2$, based on the integral equation, are formally more complicated but have the advantage that a meaningful trial function need be chosen only within the bounded potential region $U(\mathbf{r})$. Since the new forms of $\mathbf{S}_1$ and $\mathbf{S}_2$ are easily derived, they are not given explicitly here.

III. VARIATIONAL PRINCIPLES IN THE STANDARD SCATTERING PROBLEM

Variational principles for the solution of the standard scattering problem, i.e., the outgoing solution of the wave equation for an incident plane wave, follow readily by a limiting procedure from the point-source forms of the last section.

If we take the transpose of $\mathbf{S}_1(\mathbf{r},\mathbf{p})$, interchange $\mathbf{r}$ and $\mathbf{p}$, and use the reciprocity relation, Eq. (5), we obtain $\mathbf{S}_1$ in the form

$$\mathbf{S}_1(\mathbf{r},\mathbf{p})=2\mathbf{\Gamma}(\mathbf{r},\mathbf{p})+\int[\nabla'\times\nabla'\times\mathbf{\Gamma}(\mathbf{r}',\mathbf{r})-k^2\mathbf{\Gamma}(\mathbf{r}',\mathbf{r})+U(\mathbf{r}')\mathbf{\Gamma}(\mathbf{r}',\mathbf{r})]^T\mathbf{\Gamma}(\mathbf{r}',\mathbf{p})d\mathbf{r}'.\quad (11)$$

Interchanging $\mathbf{p}$ and $\mathbf{r}$, applying the reciprocity relation to the transpose of the integral equation

$$\mathbf{\Gamma}(\mathbf{r},\mathbf{p})=\mathbf{G}(\mathbf{r},\mathbf{p})+\int\mathbf{G}(\mathbf{r},\mathbf{r}')\mathbf{\Gamma}(\mathbf{r}',\mathbf{p})U(\mathbf{r}')d\mathbf{r}',\quad (12)$$

and allowing the source point $\mathbf{p}$ to recede to infinity in the direction $-\mathbf{k}_0$, we find the limiting form

$$\mathbf{\Gamma}(\mathbf{r},\mathbf{p})\to\frac{e^{ikr_p}}{-4\pi r_p}\bigg\{e^{i\mathbf{k}_0\cdot\mathbf{r}}\mathbf{I}_{\mathbf{k}_0}+\int\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')\mathbf{I}_{\mathbf{k}_0}U(\mathbf{r}')e^{i\mathbf{k}_0\cdot\mathbf{r}'}d\mathbf{r}'\bigg\}=\frac{e^{ikr_p}}{-4\pi r_p}\mathbf{A}(\mathbf{k}_0|\mathbf{r}),\quad (13)$$

where $\mathbf{A}(\mathbf{k}_0|\mathbf{r})$ is the solution to the standard scattering problem, representing the total field at the point $\mathbf{r}$ resulting from the scattering of an incident plane wave having propagation vector $\mathbf{k}_0$. If we now take the same limit in Eq. (11) and divide through by the amplitude

[10] S. Altshuler, Phys. Rev. **109**, 1830 (1958).

factor $e^{ikr_p}/(-4\pi r_p)=N$, we obtain

$$\frac{\mathbf{S}_1}{N}=2\mathbf{A}(\mathbf{k}_0|\mathbf{r})+\int[\nabla'\times\nabla'\times\mathbf{\Gamma}(\mathbf{r}',\mathbf{r})]^T\mathbf{A}(\mathbf{k}_0|\mathbf{r}')d\mathbf{r}'-\int[k^2-U(\mathbf{r}')]\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')\mathbf{A}(\mathbf{k}_0|\mathbf{r}')d\mathbf{r}'.$$

Multiplying the transpose of the differential equation (in the primed coordinates) from the right by $\mathbf{A}(\mathbf{k}_0|\mathbf{r}')$ and subtracting the differential equation for $\mathbf{A}(\mathbf{k}_0|\mathbf{r}')$, multiplied from the left by $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$, we find, after integrating over the $\mathbf{r}'$ coordinates, the identity

$$\int[\nabla'\times\nabla'\times\mathbf{\Gamma}(\mathbf{r}',\mathbf{r})]^T\mathbf{A}(\mathbf{k}_0|\mathbf{r}')d\mathbf{r}'=\int\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')\nabla'\times\nabla'\times\mathbf{A}(\mathbf{k}_0|\mathbf{r}')d\mathbf{r}'-\mathbf{A}(\mathbf{k}_0|\mathbf{r}).$$

Therefore, we have

$$\mathbf{Y}_1=\frac{\mathbf{S}_1}{N}=\mathbf{A}(\mathbf{k}_0|\mathbf{r})+\int\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')\cdot[\nabla'\times\nabla'\times-k^2+U(\mathbf{r}')]\mathbf{A}(\mathbf{k}_0|\mathbf{r}')d\mathbf{r}' \quad (14)$$

as the variational principle for the dyadic field $\mathbf{A}(\mathbf{k}_0|\mathbf{r})$. Proof of the stationarity of $\mathbf{Y}_1$ for arbitrary, independent variations of the two wave functions $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$ and $\mathbf{A}(\mathbf{k}_0|\mathbf{r})$ is deferred to Appendix II.

For the case of an incident plane wave with the electric polarization vector equal to a constant vector $\hat{e}$, the relation between the vector field $\mathbf{E}$ and the dyadic field $\mathbf{A}$ is $\mathbf{E}=\mathbf{A}\cdot\hat{e}$. Thus, scalar multiplication from the right of Eq. (14) by a polarization vector $\hat{e}$ yields a bifunctional variational principle directly on the electric field vector $\mathbf{E}$, stationary with respect to arbitrary independent variations of the two trial functions from the exact solutions to the standard vector scattering problem and the dyadic point-source problem.

The same limiting procedure of allowing the source point $\mathbf{p}$ to go to infinity in the direction $-\mathbf{k}_0$ cannot be applied directly to the stationary form $\mathbf{S}_2$ since, in this limit, the longitudinal component of the field vanishes; therefore, $\lim\mathbf{\Gamma}(\mathbf{r},\mathbf{p})=N\mathbf{A}(\mathbf{k}_0|\mathbf{r})$ is a singular matrix, and $\lim\mathbf{\Gamma}^{-1}(\mathbf{r},\mathbf{p})$ does not exist. Nevertheless, there does exist a variational principle of the exponential form for $\mathbf{A}(\mathbf{k}_0|\mathbf{r})$:

$$\mathbf{Y}_2=\left\{\exp\left[\int\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')[\nabla'\times\nabla'\times-k^2+U(\mathbf{r}')]\cdot\mathbf{A}(\mathbf{k}_0|\mathbf{r}')d\mathbf{r}'\mathbf{A}^{-1}(\mathbf{k}_0|\mathbf{r})\right]\right\}\mathbf{A}(\mathbf{k}_0|\mathbf{r}). \quad (15)$$

For exact trial functions, the exponent reduces to the zero matrix, and $\mathbf{Y}_2=\mathbf{A}(\mathbf{k}_0|\mathbf{r})$. Since $\mathbf{A}(\mathbf{k}_0|\mathbf{r})$ is singular,[11] its inverse, in the ordinary sense, does not exist; a clarification of the meaning of the quantity $\mathbf{A}^{-1}(\mathbf{k}_0|\mathbf{r})$, appearing in Eq. (15), is therefore in order.

If $\hat{e}^1$, $\hat{e}^2$ are two independent polarization vectors of the incident plane wave and if $\mathbf{E}^1(\mathbf{r})$, $\mathbf{E}^2(\mathbf{r})$ are the total electric fields at $\mathbf{r}$ due to the scattering of plane waves polarized in the directions $\hat{e}^1$, $\hat{e}^2$, respectively, then we know the function $\mathbf{A}(\mathbf{k}_0|\mathbf{r})$ has the properties

$$\begin{aligned}\mathbf{A}(\mathbf{k}_0|\mathbf{r})\hat{e}^1&=\mathbf{E}^1(\mathbf{r}),\\ \mathbf{A}(\mathbf{k}_0|\mathbf{r})\hat{e}^2&=\mathbf{E}^2(\mathbf{r}),\\ \mathbf{A}(\mathbf{k}_0|\mathbf{r})\hat{k}_0&=0,\end{aligned} \quad (16)$$

where $\hat{k}_0$ is the unit vector in the direction of propagation of the incident plane wave. Since $\mathbf{A}$ is a 3-dimensional matrix of rank 2, its inverse does not exist, although we can define a rank 2 "inverse." The third of Eqs. (16) requires that the left inverse to $\mathbf{A}$ satisfy

$$\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}_{\mathbf{k}_0}, \quad (17)$$

while the first two equations imply

$$\mathbf{A}^{-1}\mathbf{E}^1=\hat{e}^1, \quad \mathbf{A}^{-1}\mathbf{E}^2=\hat{e}^2. \quad (18)$$

These equations are not sufficient to determine $\mathbf{A}^{-1}(\mathbf{k}_0|\mathbf{r})$ uniquely, however, and another condition on $\mathbf{A}^{-1}$ is required. According to Eq. (18), $\mathbf{A}^{-1}(\mathbf{k}_0|\mathbf{r})$, operating on a solution to the plane-wave-scattering problem, yields the polarization vector of the incident plane wave. The most general solution of the scattering problem (corresponding to arbitrary polarization of the incident plane wave), at any point $\mathbf{r}$, is a vector lying in the plane defined by the two vectors $\mathbf{E}^1(\mathbf{r})$ and $\mathbf{E}^2(\mathbf{r})$. Any vector in the direction of $\mathbf{E}^1(\mathbf{r})\times\mathbf{E}^2(\mathbf{r})$ cannot result from an incident plane wave polarized in any direction; therefore, $\mathbf{A}^{-1}(\mathbf{k}_0|\mathbf{r})$ operating on $\mathbf{E}^1\times\mathbf{E}^2$ must be zero:

$$\mathbf{A}^{-1}(\mathbf{k}_0|\mathbf{r})\hat{e}^3(\mathbf{r})=0, \quad (19)$$

where $\hat{e}^3(\mathbf{r})$ is a unit vector in the direction of $\mathbf{E}^1(\mathbf{r})\times\mathbf{E}^2(\mathbf{r})$. Equations (18) and (19) define the matrix $\mathbf{A}^{-1}(\mathbf{k}_0|\mathbf{r})$ uniquely in terms of $\mathbf{E}^1$, $\mathbf{E}^2$, $\hat{e}^1$, $\hat{e}^2$. The matrix elements of $\mathbf{A}$ and $\mathbf{A}^{-1}$ may be written out explicitly; however, except for the special case in which $(\hat{e}^1, \hat{e}^2, \hat{k}_0)$ are the Cartesian basis vectors $(\hat{\imath}, \hat{\jmath}, \hat{k})$, the most concise representations are the dyadic forms

$$\mathbf{A}(\mathbf{k}_0|\mathbf{r})=(\mathbf{E}^1\hat{e}^2-\mathbf{E}^2\hat{e}^1)\times(\hat{e}^1\times\hat{e}^2)\frac{1}{(\hat{e}^1\times\hat{e}^2)^2},$$

$$\mathbf{A}^{-1}(\mathbf{k}_0|\mathbf{r})=(\hat{e}^1\mathbf{E}^2-\hat{e}^2\mathbf{E}^1)\times(\mathbf{E}^1\times\mathbf{E}^2)\frac{1}{(\mathbf{E}^1\times\mathbf{E}^2)^2}. \quad (20)$$

Finally, we note that $\hat{k}_0\mathbf{A}^{-1}=\hat{e}^3\mathbf{A}=0$, and $\mathbf{A}\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{A}=\hat{k}_0\hat{k}_0-\hat{e}^3\hat{e}^3$.

As in the case of the variational principles for the point-source wave function, the differential operator

[11] The singularity of $\mathbf{A}(\mathbf{k}_0|\mathbf{r})$ is implied by the transversality condition, $\mathbf{A}(\mathbf{k}_0|\mathbf{r})\cdot\hat{k}_0=0$.

$\nabla\times\nabla\times-k^2+U(\mathbf{r})$ may be eliminated from $\mathbf{Y}_1$ and $\mathbf{Y}_2$ by replacing the wave function by its integral equation. For example, taking the integral equation for $\mathbf{A}(\mathbf{k}_0|\mathbf{r})$ in the form

$$\mathbf{A}(\mathbf{k}_0|\mathbf{r})=\mathbf{I}_{\mathbf{k}_0}e^{i\mathbf{k}_0\cdot\mathbf{r}}+\int\mathbf{G}(\mathbf{r},\mathbf{r}')U(\mathbf{r}')\mathbf{A}(\mathbf{k}_0|\mathbf{r}')d\mathbf{r}', \quad (21)$$

and substituting in $\mathbf{Y}_1$, we obtain the stationary form

$$\begin{aligned}\mathbf{Y}_1=\mathbf{I}_{\mathbf{k}_0}e^{i\mathbf{k}_0\cdot\mathbf{r}}&+\int\mathbf{G}(\mathbf{r},\mathbf{r}')U(\mathbf{r}')\mathbf{A}(\mathbf{k}_0|\mathbf{r}')d\mathbf{r}'\\&+\int\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}')\mathbf{I}_{\mathbf{k}_0}e^{i\mathbf{k}_0\cdot\mathbf{r}'}-\int\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}')U(\mathbf{r}')\mathbf{A}(\mathbf{k}_0|\mathbf{r}')d\mathbf{r}'\\&+\int\int\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}')U(\mathbf{r}')\mathbf{G}(\mathbf{r}',\mathbf{r}'')U(\mathbf{r}'')\\&\qquad\cdot\mathbf{A}(\mathbf{k}_0|\mathbf{r}'')d\mathbf{r}'d\mathbf{r}''. \quad (22)\end{aligned}$$

If $\mathbf{I}_{\mathbf{k}_0}e^{i\mathbf{k}_0\cdot\mathbf{r}}$ is subtracted from $\mathbf{Y}_1$ in either of the forms (14) or (22), there results a variational principle for the scattered field; if then the observation point $\mathbf{r}$ is allowed to become infinitely large in the direction $\hat{k}$, there follows a principle for the tensor amplitude $\mathbf{F}(\mathbf{k}_0|\mathbf{k})$ for scattering into the direction given by the unit vector $\hat{k}$. Thus, from Eq. (14), we have

$$\begin{aligned}\frac{e^{ikr}}{r}[\mathbf{F}(\mathbf{k}_0|\mathbf{k})]&=\lim_{\hat{k}r\to\infty}(\mathbf{Y}_1-\mathbf{I}_{\mathbf{k}_0}e^{i\mathbf{k}_0\cdot\mathbf{r}})\\&=\frac{e^{ikr}}{r}\mathbf{F}(\mathbf{k}_0|\mathbf{k})+\frac{e^{ikr}}{-4\pi r}\int\mathbf{A}^T(-\mathbf{k}|\mathbf{r}')\\&\qquad\cdot[\nabla'\times\nabla'\times-k^2+U(\mathbf{r}')]\mathbf{A}(\mathbf{k}_0|\mathbf{r}')d\mathbf{r}',\end{aligned}$$

or

$$\begin{aligned}[\mathbf{F}(\mathbf{k}_0|\mathbf{k})]=\mathbf{F}(\mathbf{k}_0|\mathbf{k})-\frac{1}{4\pi}\int\mathbf{A}^T(-\mathbf{k}|\mathbf{r}')\\\cdot[\nabla'\times\nabla'\times-k^2+U(\mathbf{r}')]\mathbf{A}(\mathbf{k}_0|\mathbf{r}')d\mathbf{r}'. \quad (23)\end{aligned}$$

This is the dyadic analog of the Kohn[12] bifunctional variational principle for the scalar scattering amplitude. As may be readily established from the two equivalent forms of the integral equation for $\mathbf{A}(\mathbf{k}_0|\mathbf{r})$, Eqs. (9) and (21), the scattering amplitude satisfies the transversality conditions $\mathbf{F}(\mathbf{k}_0|\mathbf{k})\mathbf{k}_0=\mathbf{k}\mathbf{F}(\mathbf{k}_0|\mathbf{k})=0$, and the reciprocity relation $\mathbf{F}(\mathbf{k}_0|\mathbf{k})=\mathbf{F}^T(-\mathbf{k}|-\mathbf{k}_0)$.

Finally, there also exists an amplitude-independent variational principle for the dyadic scattering amplitude which is the formal generalization of Schwinger's well-known variational principle for the scalar transition amplitude; that is,

$$\begin{aligned}[\mathbf{F}(\mathbf{k}_0|\mathbf{k})]=\left(\frac{-1}{4\pi}\right)\int\mathbf{I}_k\mathbf{A}(\mathbf{k}_0|\mathbf{r}')U(\mathbf{r}')e^{-i\mathbf{k}\cdot\mathbf{r}'}d\mathbf{r}'\cdot\mathbf{C}^{-1}\\\cdot\int\mathbf{A}^T(-\mathbf{k}|\mathbf{r}')\mathbf{I}_{\mathbf{k}_0}U(\mathbf{r}')e^{i\mathbf{k}_0\cdot\mathbf{r}'}d\mathbf{r}', \quad (24)\end{aligned}$$

where $\mathbf{C}$ is the rank 2 matrix

$$\begin{aligned}\mathbf{C}\equiv\int\mathbf{A}^T(-\mathbf{k}|\mathbf{r}')\mathbf{A}(\mathbf{k}_0|\mathbf{r}')U(\mathbf{r}')d\mathbf{r}'-\int\int\mathbf{A}^T(-\mathbf{k}|\mathbf{r}')\\\cdot\mathbf{G}(\mathbf{r}',\mathbf{r}'')\mathbf{A}(\mathbf{k}_0|\mathbf{r}'')U(\mathbf{r}')U(\mathbf{r}'')d\mathbf{r}'d\mathbf{r}''\end{aligned}$$

which reduces to $\mathbf{F}(\mathbf{k}_0|\mathbf{k})$ for exact trial functions, $\mathbf{A}(\mathbf{k}_0|\mathbf{r}')$, $\mathbf{A}^T(-\mathbf{k}|\mathbf{r}')$. The "inverse," $\mathbf{C}^{-1}$, of the matrix $\mathbf{C}$ is defined in a manner analogous to the definition of $\mathbf{A}^{-1}$. If $\mathfrak{J}$ represents the integral operator

$$\begin{aligned}\mathfrak{J}\equiv\int d\mathbf{r}'\mathbf{A}^T(-\mathbf{k}|\mathbf{r}')U(\mathbf{r}')\\-\int\int d\mathbf{r}'d\mathbf{r}''\mathbf{A}^T(-\mathbf{k}|\mathbf{r}')\mathbf{G}(\mathbf{r}',\mathbf{r}'')U(\mathbf{r}')U(\mathbf{r}''),\end{aligned}$$

and $\mathbf{E}^1(\mathbf{r})$ and $\mathbf{E}^2(\mathbf{r})$ are the trial electric fields associated with the trial wave function $\mathbf{A}(\mathbf{k}_0|\mathbf{r})$, then the following properties obtain

$$\begin{aligned}\mathbf{C}\hat{e}^1&=\mathfrak{J}\mathbf{E}^1,\\\mathbf{C}\hat{e}^2&=\mathfrak{J}\mathbf{E}^2,\\\mathbf{C}\hat{k}_0&=0,\end{aligned}\quad (25)$$

and

$$\begin{aligned}\mathbf{C}^{-1}(\mathfrak{J}\mathbf{E}^1)&=\hat{e}^1,\\\mathbf{C}^{-1}(\mathfrak{J}\mathbf{E}^2)&=\hat{e}^2,\\\mathbf{C}^{-1}(\mathfrak{J}\mathbf{E}^1\times\mathfrak{J}\mathbf{E}^2)&=0.\end{aligned}\quad (26)$$

Furthermore, $\hat{k}\mathbf{C}=\hat{k}_0\mathbf{C}^{-1}=0$, $\mathbf{C}^{-1}\mathbf{C}=\mathbf{I}_{\mathbf{k}_0}$, $\mathbf{C}\mathbf{C}^{-1}=\mathbf{I}-\hat{k}\hat{k}$, and, since $\hat{k}\mathfrak{J}\mathbf{E}^1=\hat{k}\mathfrak{J}\mathbf{E}^2=0$, it follows that the vector $\mathfrak{J}\mathbf{E}^1\times\mathfrak{J}\mathbf{E}^2$ is parallel to $\pm\hat{k}$. The elements of the matrix $\mathbf{C}^{-1}$ may be determined algebraically from Eqs. (26); alternatively, the dyadic representation of Eq. (20) may be used, with $\mathbf{E}^1$ and $\mathbf{E}^2$ replaced by $\mathfrak{J}\mathbf{E}^1$ and $\mathfrak{J}\mathbf{E}^2$, respectively.

IV. SPHERICALLY SYMMETRIC SCATTERER

It is of interest to consider the special case of a spherically symmetric potential, $U(\mathbf{r})=U(r)$, for in this case, the expansion of the field into vector spherical waves leads to a relatively simple representation of the solution in terms of one-dimensional integral equations

[12] W. Kohn, Phys. Rev. 74, 1763 (1948).

for the radial parts of the component spherical waves. Comparison of the asymptotic form of the total field with that of the unperturbed field leads to the introduction of phase shifts, of which two are required for each spherical wave component of the total field, in contrast to the single phase shift necessary in the scalar problem. Variational principles will be given for the cotangent of each of the two phase shifts.

It is convenient to expand the field into vector spherical harmonics, $\mathbf{Y}_{jl}{}^m(\theta,\varphi)$, although for the spin-one electromagnetic field only the three functions $\mathbf{Y}_{j,j}{}^m$, $\mathbf{Y}_{j,j-1}{}^m$, and $\mathbf{Y}_{j,j+1}{}^m$ are required. These functions are related to the more familiar scalar complex spherical harmonics, $Y_j{}^m(\theta,\varphi)$, by the equations

$$\mathbf{Y}_{j,j}{}^m=[j(j+1)]^{-1/2}\mathbf{L}Y_j{}^m,$$
$$\mathbf{Y}_{j,j-1}{}^m=[j(2j+1)]^{-1/2}[-j\hat{r}+i\hat{r}\times\mathbf{L}]Y_j{}^m,$$
$$\mathbf{Y}_{j,j+1}{}^m=[(j+1)(2j+1)]^{-1/2}[(j+1)\hat{r}+i\hat{r}\times\mathbf{L}]Y_j{}^m,$$

where $\mathbf{L}$ is the operator $-i\mathbf{r}\times\nabla$, and j, m are integers. The vector harmonics are normalized by the condition

$$\int \mathbf{Y}_{jl}{}^{m*}(\theta,\varphi)\cdot\mathbf{Y}_{j'l'}{}^{m'}(\theta,\varphi)d\Omega=\delta_{jj'}\delta_{ll'}\delta_{mm'}.$$

In terms of the $\mathbf{Y}_{jl}{}^m$, the dyadic Green's function, Eq. (8), may be shown to have the expansion

$$\mathbf{G}(\mathbf{r},\mathbf{r}')=-ik\sum_{jlm} j_l(kr')h_l(kr)\mathbf{Y}_{jl}{}^m(\theta,\varphi)\mathbf{Y}_{jl}{}^{m*}(\theta',\varphi')+ik\sum_{jm}\left[\left(\frac{j}{2j+1}\right)^{1/2}h_{j-1}(kr)\mathbf{Y}_{jj-1}{}^m(\theta,\varphi)\right.$$
$$\left.+\left(\frac{j+1}{2j+1}\right)^{1/2}h_{j+1}(kr)\mathbf{Y}_{jj+1}{}^m(\theta,\varphi)\right]\left[\left(\frac{j}{2j+1}\right)^{1/2}j_{j-1}(kr')\mathbf{Y}_{jj-1}{}^m(\theta',\varphi')+\left(\frac{j+1}{2j+1}\right)^{1/2}j_{j+1}(kr')\mathbf{Y}_{jj+1}{}^m(\theta',\varphi')\right] \quad (27)$$

for $r>r'$; for $r<r'$, $\mathbf{G}(\mathbf{r},\mathbf{r}')$ is found from Eq. (27) by interchanging primed and unprimed coordinates and transposing. The range of summation on j is from 1 to ∞, on l from $j-1$ to $j+1$, and on m from $-j$ to $+j$. The functions $j_l(kr)$ and $h_l(kr)$ are spherical Bessel functions of order l and spherical Hankel functions of the second kind of order l, respectively. An incident plane wave, propagating in the z direction and polarized in the x direction, has the expansion

$$\mathbf{E}_0=Ae^{ikz}\hat{x}$$
$$=\sum_{l,j}-A(2\pi)^{1/2}(2l+1)^{1/2}i^lj_l(kr)$$
$$\times(C_{01}{}^{l1j}\mathbf{Y}_{jl}{}^1-C_{0-1}{}^{l1j}\mathbf{Y}_{jl}{}^{-1}), \quad (28)$$

where the $C_{0\pm1l}{}^{1j}$ are 5-index Clebsch-Gordan coefficients, and A is the amplitude of the incident wave.

If we write for the total electric field

$$\mathbf{E}(\mathbf{r})=\sum_{jlm} w_{jl}{}^m(r)\mathbf{Y}_{jl}{}^m(\theta,\varphi) \quad (29)$$

and substitute Eqs. (27)–(29) into the integral equation for $\mathbf{E}(\mathbf{r})$

$$\mathbf{E}(\mathbf{r})=\mathbf{E}_0(\mathbf{r})+\int U(r')\mathbf{G}(\mathbf{r},\mathbf{r}')\mathbf{E}(\mathbf{r}')r'^2dr'd\Omega',$$

we obtain an expression in which the angular integration over $d\Omega'$ can be carried out. A further application of the orthonormality property of the $\mathbf{Y}_{jl}{}^m$ then yields the following three purely radial integral equations for the unknown functions $w_{jj}{}^m(r)$, $w_{jj-1}{}^m(r)$, and $w_{jj+1}{}^m(r)$:

$$w_{jj}{}^m(r)=[\pi(2j+1)]^{1/2}i^jj_j(kr)(\delta_{m,1}+\delta_{m,-1})A-ikh_j(kr)\int_0^r r'^2U(r')j_j(kr')w_{jj}{}^m(r')dr'$$
$$-ikj_j(kr)\int_r^\infty r'^2U(r')h_j(kr')w_{jj}{}^m(r')dr', \quad (30)$$

$$w_{jj-1}{}^m(r)=-[\pi(j+1)]^{1/2}i^{j-1}j_{j-1}(kr)(\delta_{m,1}-\delta_{m,-1})A-ik\frac{(j+1)^{1/2}}{2j+1}h_{j-1}(kr)\int_0^r r'^2U[(j+1)^{1/2}j_{j-1}w_{jj-1}{}^m$$
$$-(j)^{1/2}j_{j+1}w_{jj+1}{}^m]dr'-ik\frac{(j+1)^{1/2}}{2j+1}j_{j-1}(kr)\int_r^\infty r'^2U[(j+1)^{1/2}h_{j-1}w_{jj-1}{}^m-(j)^{1/2}h_{j+1}w_{jj+1}{}^m]dr', \quad (31)$$

$$w_{jj+1}{}^m(r)=-(\pi j)^{1/2}i^{j+1}j_{j+1}(kr)(\delta_{m,1}-\delta_{m,-1})A-ik\frac{(j)^{1/2}}{2j+1}h_{j+1}(kr)\int_0^r r'^2U[(j)^{1/2}j_{j+1}w_{jj+1}{}^m$$
$$-(j+1)^{1/2}j_{j-1}w_{jj-1}{}^m]dr'-\frac{ik(j)^{1/2}}{2j+1}j_{j+1}(kr)\int_r^\infty r'^2U[(j)^{1/2}h_{j+1}w_{jj+1}{}^m-(j+1)^{1/2}h_{j-1}w_{jj-1}{}^m]dr'. \quad (32)$$

It is readily shown from these equations that nontrivial solutions are possible only for $m=\pm 1$; furthermore, it is clear that

$$w_{jj}{}^{-1}=w_{jj}{}^{1},\quad w_{jj\pm 1}{}^{-1}=-w_{jj\pm 1}{}^{1}.$$

Since the potential is assumed to be bounded, $U(r)=0$ for all r greater than some distance a; thus, for $r>a$, the last term in each of the radial equations is zero. In terms of conveniently redefined radial functions (which are independent of whether $m=\pm 1$)

$$v_j(r)=\frac{w_{jj}{}^m(r)kr}{(\pi)^{1/2}i^j},\quad v_{j-1}(r)=\frac{w_{jj-1}{}^m(r)kr}{-(\pi)^{1/2}i^{j-1}(\delta_{m,1}-\delta_{m,-1})},\quad v_{j+1}(r)=\frac{w_{jj+1}{}^m(r)kr}{-(\pi)^{1/2}i^{j-1}(\delta_{m,1}-\delta_{m,-1})},\tag{33}$$

the system of coupled radial integral equations simplifies, for large r, to

$$v_j(r)=(2j+1)^{1/2}krj_j(kr)A-ikrh_j(kr)\int_0^r U(r')j_j(kr')v_j(r')r'dr',$$

$$v_{j-1}(r)=(j+1)^{1/2}krj_{j-1}(kr)A-ikrh_{j-1}(kr)\frac{(j+1)^{1/2}}{2j+1}\int_0^r[(j+1)^{1/2}j_{j-1}v_{j-1}+(j)^{1/2}j_{j+1}v_{j+1}]Ur'dr',\tag{34}$$

$$v_{j+1}(r)=(j)^{1/2}krj_{j+1}(kr)A-ikrh_{j+1}(kr)\frac{(j)^{1/2}}{2j+1}\int_0^r[(j+1)^{1/2}j_{j-1}v_{j-1}+(j)^{1/2}j_{j+1}v_{j+1}]Ur'dr'.$$

Letting $r\rightarrow\infty$, we obtain the asymptotic form of the radial functions

$$v_j(r)=(2j+1)^{1/2}A\cos\left[kr-\frac{\pi}{1}(j+1)\right]-e^{ikr-i(\pi/2)(j+1)}\int_0^\infty iU(r')j_j(kr')v_j(r')r'dr',$$

$$v_{j-1}(r)=(j+1)^{1/2}A\cos\left[kr-\frac{\pi}{2}j\right]-e^{ikr-i(\pi/2)j}\int_0^\infty\frac{iU(r')}{2j+1}\{(j+1)j_{j-1}v_{j-1}+[j(j+1)]^{1/2}j_{j+1}v_{j+1}\}r'dr',\tag{35}$$

$$v_{j+1}(r)=-(j)^{1/2}A\cos\left[kr-\frac{\pi}{2}j\right]+e^{ikr-i(\pi/2)j}\int_0^\infty\frac{iU(r')}{2j+1}\{[j(j+1)]^{1/2}j_{j-1}v_{j-1}+jj_{j+1}v_{j+1}\}r'dr'.$$

Note that asymptotically there are only two independent radial functions, since, according to Eq. (35), $v_{j+1}=-[j/(j+1)]^{1/2}v_{j-1}$. The spherical wave components of the incident plus scattered waves may be interpreted asymptotically as the components of a phase-shifted incident wave. Thus, defining the jth-order phase shifts α_j, β_j by

$$v_{j-1}=B_j(j+1)^{1/2}\cos[kr-(\pi/2)j-\alpha_j],\quad v_j=C_j(2j+1)^{1/2}\cos[kr-(\pi/2)(j+1)-\beta_j],\tag{36}$$

we find $B_j=Ae^{-i\alpha_j}$, $C_j=Ae^{-i\beta_j}$, and

$$\tan\alpha_j=\frac{1}{A}\int_0^\infty rU(r)\left[\left(\frac{j+1}{2j+1}\right)^{1/2}j_{j-1}(kr)\varphi_{j-1}(r)+\left(\frac{j}{2j+1}\right)^{1/2}j_{j+1}(kr)\varphi_{j+1}(r)\right]dr,\tag{37}$$

$$\tan\beta_j=\frac{1}{A}\int_0^\infty rU(r)j_j(kr)\varphi_j(r)dr,$$

where

$$\varphi_j(r)=\frac{e^{i\beta_j}v_j(r)}{(2j+1)^{1/2}\cos\beta_j},\quad \varphi_{j\pm 1}(r)=\frac{e^{i\alpha_j}v_{j\pm 1}(r)}{(2j+1)^{1/2}\cos\alpha_j}.$$

If the integral equation for $v_j(r)$, expressed in terms of the functions $\varphi_j(r)$, is multiplied by $kU(r)\varphi_j(r)$ and integrated on r from 0 to ∞, it is found, after some manipulation, that

$$\cot\beta_j=+k\left(\int U(r)[\varphi_j(r)]^2dr+\int dr\int dr'\varphi_j(r)U(r)g_j(r,r')U(r')\varphi_j(r')\right)\Big/\left[\int krU(r)j_j(kr)\varphi_j(r)dr\right]^2,\tag{38}$$

where

$$g_j(r,r') = -krr'j_j(kr)n_j(kr'), \quad r<r'$$
$$= -kr'rj_j(kr')n_j(kr), \quad r>r'$$

and $n_j(r)$ is the spherical Neumann function of order j. This equation for $\cot\beta_j$ is stationary with respect to small variations of a trial function $\varphi_j(r)$ from the true wave function. In fact, Eq. (38) is identical to a well-known amplitude-independent variational principle for the phase shift in the scalar theory[13]; this is to be expected, of course, since the radial wave function, $w_{jj}{}^m(r)$, satisfies the same integral equation as the jth radial function in the partial wave expansion of the scalar theory.

The development of a variational principle for the second phase shift, α_j, characteristic of the vector problem is carried out by a method much the same as that leading to Eq. (38). However, because of the two functions, $\varphi_{j-1}(r)$ and $\varphi_{j+1}(r)$, which now appear, it will be convenient to introduce a single 2-component vector function

$$\boldsymbol{\varphi}(r) = \begin{pmatrix} \varphi_{j+1}(r) \\ \varphi_{j-1}(r) \end{pmatrix};$$

in addition, we shall define the vectors

$$\mathbf{J}(r) = \frac{kr}{(2j+1)^{1/2}} \begin{pmatrix} (j+1)^{1/2} j_{j-1}(kr) \\ (j)^{1/2} j_{j+1}(kr) \end{pmatrix},$$

$$\mathbf{H}(r) = \frac{kr}{(2j+1)^{1/2}} \begin{pmatrix} (j+1)^{1/2} h_{j-1}(kr) \\ (j)^{1/2} h_{j+1}(kr) \end{pmatrix} = \mathbf{J}(r) + i\mathbf{N}(r),$$

and the dyadic

$$\mathfrak{g}(r,r') = -\frac{1}{k} \begin{cases} \mathbf{J}(r)\mathbf{N}(r'), & r<r' \\ \mathbf{N}(r)\mathbf{J}(r'), & r>r' \end{cases}.$$

Equations (31) and (32) may then be written as a single vector equation

$$\boldsymbol{\varphi}(r) = \cot\alpha_j \mathbf{J}(r) \frac{1}{k} \int_0^\infty U(r')\mathbf{J}(r')\cdot\boldsymbol{\varphi}(r')dr' - \int_0^\infty \mathfrak{g}(r,r')\cdot\boldsymbol{\varphi}(r')U(r')dr', \quad (39)$$

where the amplitude A has been eliminated by using the first of Eq. (37), i.e.,

$$A = \frac{1}{k}\cot\alpha_j \int_0^\infty U(r')\mathbf{J}(r')\cdot\boldsymbol{\varphi}(r')dr'.$$

Scalar multiplication of $\boldsymbol{\varphi}(r)$ by $\int dr\, U(r)\boldsymbol{\varphi}(r)$ then leads to the equation

$$\cot\alpha_j = +k\left(\int_0^\infty U(r)[\boldsymbol{\varphi}(r)]^2 dr + \int_0^\infty dr \int_0^\infty dr' U(r)\boldsymbol{\varphi}(r)\cdot\mathfrak{g}(r,r')\cdot\boldsymbol{\varphi}(r')U(r')\right) \Big/ \left[\int_0^\infty U(r)\mathbf{J}(r)\cdot\boldsymbol{\varphi}(r)dr\right]^2, \quad (40)$$

where $[\boldsymbol{\varphi}(r)]^2 = \boldsymbol{\varphi}\cdot\boldsymbol{\varphi} = (\varphi_{j-1})^2 + (\varphi_{j+1})^2$. The formal similarity of Eq. (40) with the variational principle for $\cot\beta_j$ is immediately evident. This expression for $\cot\alpha_j$ does, indeed, have the stationary property for arbitrary choice of the trial function $\boldsymbol{\varphi}(r)$, which may be shown as follows:

Letting the denominator in Eq. (40) be D^2, we have

$$\delta\left[\frac{1}{k}\cot\alpha_j\right] = D^{-2}\left[2\int U\boldsymbol{\varphi}\cdot\delta\boldsymbol{\varphi}dr + \int\int U\delta\boldsymbol{\varphi}(r)\cdot\mathfrak{g}\cdot\boldsymbol{\varphi}(r')U(r')drdr' + \int\int U\boldsymbol{\varphi}(r)\cdot\delta\boldsymbol{\varphi}(r')Udrdr'\right] - \frac{\cot\alpha_j}{k}2D^{-1}\int U\mathbf{J}\cdot\delta\boldsymbol{\varphi}dr,$$

where the arguments of functions are suppressed if there can be no confusion resulting therefrom. Replacing $\boldsymbol{\varphi}(r)$ in the first integral by Eq. (39) leads to

$$\delta\left[\frac{1}{k}\cot\alpha_j\right] = D^{-2}\left\{-\int\int U(r)\delta\boldsymbol{\varphi}(r)\cdot\mathfrak{g}(r,r')\cdot\boldsymbol{\varphi}(r')U(r')drdr' + \int\int U(r)\boldsymbol{\varphi}(r)\cdot\mathfrak{g}(r,r')\cdot\delta\boldsymbol{\varphi}(r')U(r')drdr'\right\}.$$

The first integral may be rewritten as

$$\int\int U(r)\delta\boldsymbol{\varphi}(r)\cdot\mathfrak{g}(r,r')\cdot\boldsymbol{\varphi}(r')U(r')drdr' = \int\int U(r')\delta\boldsymbol{\varphi}(r')\cdot\mathfrak{g}(r',r)\cdot\boldsymbol{\varphi}(r)U(r)dr'dr$$
$$= \int\int U(r')\delta\boldsymbol{\varphi}(r')\cdot\mathfrak{g}^T(r,r')\cdot\boldsymbol{\varphi}(r)U(r)dr'dr$$
$$= \int\int U(r')\boldsymbol{\varphi}(r)\cdot\mathfrak{g}(r,r')\cdot\delta\boldsymbol{\varphi}(r')U(r)dr'dr,$$

[13] P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), p. 1127.

since

$$\delta\varphi(r')\cdot\mathfrak{g}^T(r,r')\cdot\varphi(r)=\varphi(r)\cdot\mathfrak{g}(r,r')\cdot\delta\varphi(r');$$

and, therefore,

$$\delta\left[\frac{1}{k}\cot\alpha_j\right]=0.$$

V. AXIALLY SYMMETRIC POTENTIAL

The application of the variational methods developed in the previous sections are illustrated by calculating the electric field in the neighborhood of an axially symmetric scattering region, resulting from a high-energy plane wave incident along the axis of symmetry. The spatial variation of the scattering potential is described by

$$U(\mathbf{r})=u_0k^2e^{-(x^2+y^2)/2l^2}e^{-|z|/L},\quad z\leq 0$$
$$=0,\quad z>0,$$

where u_0 may be complex, in general. It is assumed that the plane wave is incident from the $-z$ direction and polarized in the y direction, and that $l\sim L$, $kl\gg 1$, and $|U|\ll k^2$; in addition, the observation point is assumed to lie in the interior of the scattering region ($z<0$). For wavelengths small relative to the scale lengths of the medium it might be expected that the near-field could be described approximately by a phase-corrected plane wave. It has been shown by Altshuler,[10] for the general scalar problem, that the simplest choice of trial functions in the scalar exponential variational principle leads to an approximate wave function, identical to that derived in different ways by Rytov[14] and Obukhov,[15] which contains diffraction effects and which reduces, for very large k, and small $\mathbf{r}$, to the well-known eikonal solution. This suggests that the dyadic exponential variational principle, Eq. (15), might also provide a reasonable approximation to the true vector field. Accordingly, we shall choose as trial functions in $\mathbf{Y}_2$ the unperturbed plane wave, $\mathbf{I}_{k_0}e^{i\mathbf{k}_0\cdot\mathbf{r}}$, and the free-space Green's function, $\mathbf{G}(\mathbf{r},\mathbf{r}')$, for $\mathbf{A}(\mathbf{k}_0|\mathbf{r})$ and $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$, respectively.

The matrix $\mathbf{A}^{-1}(\mathbf{k}_0|\mathbf{r})$ is then

$$\mathbf{A}^{-1}(\mathbf{k}_0|\mathbf{r})=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}e^{-i\mathbf{k}_0\cdot\mathbf{r}},$$

and the variational expression assumes the form

$$\mathbf{Y}=\left\{\exp\left[\int\mathbf{G}(\mathbf{r},\mathbf{r}')\mathbf{I}_{k_0}e^{i\mathbf{k}_0\cdot\mathbf{r}'}U(\mathbf{r}')d\mathbf{r}'e^{-i\mathbf{k}_0\cdot\mathbf{r}}\right]\right\}\mathbf{I}_{k_0}e^{i\mathbf{k}_0\cdot\mathbf{r}}.\tag{41}$$

Using $\mathbf{G}(\mathbf{r},\mathbf{r}')=(\mathbf{I}+k^{-2}\nabla\nabla)g(\mathbf{r},\mathbf{r}')$, the exponent may be written as $e^{-i\mathbf{k}_0\cdot\mathbf{r}}[(\mathbf{I}+k^{-2}\nabla\nabla)J(\mathbf{r})]\cdot\mathbf{I}_{k_0}$, where

$$J(\mathbf{r})=\int g(\mathbf{r},\mathbf{r}')e^{i\mathbf{k}_0\cdot\mathbf{r}'}U(\mathbf{r}')d\mathbf{r}'.$$

This integral has been evaluated for small r, through terms of order $1/k^2$, by Schiff,[16] using the method of stationary phase:

$$J(\mathbf{r})\cong-\frac{i}{2k}e^{i\mathbf{k}_0\cdot\mathbf{r}}\int_{-\infty}^{z}U(x,y,s)ds+\frac{1}{4k^2}e^{i\mathbf{k}_0\cdot\mathbf{r}}\times\int_0^\infty z'[\nabla'^2U(x-x',\,y-y',\,z-z')]_{x'=y'=0}dz'.$$

Substituting this result into the exponent, we obtain for $\mathbf{Y}$ the expression

$$\mathbf{Y}=\exp(\mathbf{M}\alpha)\mathbf{I}_{k_0}e^{i\mathbf{k}_0\cdot\mathbf{r}},$$

where

$$\alpha=\frac{-i}{2k}\int_{-\infty}^{z}dsU(x,y,s)=\frac{-iLU(x,y,z)}{2k}$$

and the elements of the matrix M are, through terms of order $(kl)^{-2}$,

$$M_{11}=1+\left(\frac{iL}{2k}\frac{1}{L^2}+\frac{x^2+y^2-2l^2}{l^4}\right)-\frac{1}{k^2l^2}\left(1-\frac{x^2}{l^2}\right),$$

$$M_{12}=M_{21}=\frac{xy}{k^2l^4},$$

$$M_{22}=1+\frac{iL}{2k}\left(\frac{1}{L^2}+\frac{x^2+y^2-2l^2}{l^4}\right)-\frac{1}{k^2l^2}\left(1-\frac{y^2}{l^2}\right),$$

$$M_{31}=\frac{-ix}{kl^2}-\frac{xL}{2k^2l^2}\left(\frac{1}{L^2}-\frac{x^2+y^2-4l^2}{l^4}\right),$$

$$M_{32}=\frac{-iy}{kl^2}-\frac{yL}{2k^2l^2}\left(\frac{1}{L^2}-\frac{x^2+y^2-4l^2}{l^4}\right),$$

$$M_{13}=M_{23}=M_{33}=0.$$

In order to sum the infinite series of matrices represented by the exponential $\exp(\mathbf{M}\alpha)$, it is sufficient to diagonalize the 2×2 submatrix formed by the first 2 rows and columns of $\mathbf{M}$; that is

$$\mathbf{M}'=\mathbf{TMT}^{-1}=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\\lambda_3&\lambda_4&0\end{bmatrix}.$$

The orthogonal matrix $\mathbf{T}$ is found to be

$$\mathbf{T}=\begin{bmatrix}x/\rho&y/\rho&0\\-y/\rho&x/\rho&0\\0&0&1\end{bmatrix},\quad\rho=(x^2+y^2)^{1/2}.$$

[14] S. M. Rytov, Izv. Akad. Nauk. S.S.S.R. Ser. Fiz. **2**, 229 (1936).

[15] A. M. Obukhov, Izv. Akad. Nauk. S.S.S.R. Ser. Geofiz. **1953**, 155 (1953) (translation by M. D. Friedman).

[16] L. I. Schiff, Phys. Rev. **103**, 443 (1956).

The transformation $\mathbf{T}$ applied to $\mathbf{Y}$, $\mathbf{Y}'=\mathbf{TYT}^{-1}$, results in an expression for $\mathbf{Y}'$ which is then easily summed to the value[17]

$$\mathbf{Y}'=\begin{Bmatrix} e^{\lambda_1\alpha} & 0 & 0 \\ 0 & e^{\lambda_2\alpha} & 0 \\ \frac{\lambda_3}{\lambda_1}(e^{\lambda_1\alpha}-1) & \frac{\lambda_4}{\lambda_2}(e^{\lambda_2\alpha}-1) & 0 \end{Bmatrix} e^{i\mathbf{k}_0\cdot\mathbf{r}},$$

in which

$$\lambda_1=1+\frac{iL}{2k}\left(\frac{1}{L^2}+\frac{x^2+y^2-2l^2}{l^4}\right)-\frac{1}{k^2l^2}\left(1-\frac{x^2+y^2}{l^2}\right),$$

$$\lambda_2=1+\frac{iL}{2k}\left(\frac{1}{L^2}+\frac{x^2+y^2-2l^2}{l^4}\right)-\frac{1}{k^2l^2},$$

$$\lambda_3=\rho^{-1}(xM_{31}+yM_{32}),$$

$$\lambda_4=\rho^{-1}(xM_{32}-yM_{31}).$$

Transforming back to the original coordinate system and using $\mathbf{E}=\mathbf{Y}\cdot\hat{\jmath}$, yields the variationally improved estimate for the electric field interior to the scattering region:

$$E_x=xy\rho^{-2}(e^{\lambda_1\alpha}-e^{\lambda_2\alpha}),$$

$$E_y=\rho^{-2}(y^2e^{\lambda_1\alpha}+x^2e^{\lambda_2\alpha}),$$

$$E_z=\frac{\lambda_3 y}{\lambda_1\rho}(e^{\lambda_1\alpha}-1)+\frac{\lambda_4 x}{\lambda_2\rho}(e^{\lambda_2\alpha}-1).$$

Simplifying these equations by retaining only terms through order k^{-2}, leads to the final results

$$E_x=\frac{xy}{k^2l^4}\alpha e^{i\mathbf{k}_0\cdot\mathbf{r}+\alpha},$$

$$E_y=e^{i\mathbf{k}_0\cdot\mathbf{r}+\alpha}+\left[\frac{iL}{2k}\left(\frac{1}{L^2}+\frac{x^2+y^2-2l^2}{l^4}\right)-\frac{1}{k^2l^2}\left(1-\frac{y^2}{l^2}\right)\right]\times\alpha e^{i\mathbf{k}_0\cdot\mathbf{r}+\alpha}-\frac{L^2}{8k^2}\left(\frac{1}{L^2}+\frac{x^2+y^2-2l^2}{l^4}\right)^2\alpha^2 e^{i\mathbf{k}_0\cdot\mathbf{r}+\alpha},$$

$$E_z=\frac{-iy}{kl^2}e^{i\mathbf{k}_0\cdot\mathbf{r}}\left\{(e^\alpha-1)\left[1-\frac{i}{kL}\left(1+\frac{L^2}{l^2}\right)\right]+e^\alpha\frac{iL}{2k}\left(\frac{1}{L^2}+\frac{x^2+y^2-2l^2}{l^4}\right)\right\}.$$

These equations are consistent with those obtained earlier[18] by using a phase-modified plane wave as a first approximation in the right-hand side of the integral equation

$$\mathbf{E}(\mathbf{r})=e^{i\mathbf{k}_0\cdot\mathbf{r}}\hat{\jmath}+\int g(\mathbf{r},\mathbf{r}')\left\{U(\mathbf{r}')\mathbf{E}(\mathbf{r}')+\nabla\left(\frac{\nabla U\cdot\mathbf{E}}{k^2-U}\right)\right\}d\mathbf{r}'.$$

Although the potential is restricted by $|U|\ll k^2$, the phase correction α is not required to be small; in fact, the condition on the potential implies only that $|\alpha|\ll kL$, where $kL\gg1$. If the medium is highly absorbing, α will have a large imaginary part, and $|e^{i\mathbf{k}_0\cdot\mathbf{r}+\alpha}|=e^{-\beta}$, where β is the total amplitude attenuation. For sufficiently large β all terms in the field components are exponentially damped, with the exception of one term in E_z; thus

$$\mathbf{E}\approx\frac{iy}{kl^2}e^{i\mathbf{k}_0\cdot\mathbf{r}}\hat{z},$$

and the internal field apparently is predominantly longitudinally polarized. This result is open to question, however, since it may be argued that a plane-wave trial solution in the presence of a highly absorbent medium is not sufficiently realistic for the variational method to yield an improved wave function.[19]

It is interesting to compare Eq. (41) with an approximate solution of the scalar scattering problem. Rytov and Obukhov have shown that the wave function

$$\psi(\mathbf{k}_0|\mathbf{r})=e^{i\mathbf{k}_0\cdot\mathbf{r}}\exp\left(\int\frac{\exp[ik|\mathbf{r}-\mathbf{r}'|+i\mathbf{k}_0\cdot(\mathbf{r}'-\mathbf{r})]}{-4\pi|\mathbf{r}-\mathbf{r}'|}U(\mathbf{r}')d\mathbf{r}'\right) \tag{42}$$

provides, under certain conditions, a good approximation to the exact solution of the scalar wave equation; the same wave function was shown by Altshuler[10] to result directly from an exponential form of variational principle. Equation (41) obviously is precisely analogous to Eq. (42), with the scalar functions being simply replaced by their dyadic equivalents; this correspondence suggests that the dyadic function, Eq. (41), is the generalization of the scalar Rytov-Obukhov approximation to the vector-scattering problem. Substitution of the approximate solution into the differential equation in order to determine conditions of validity of the approximate wave function cannot be easily carried out in the vector problem, in contrast to the scalar problem, since the presence of the dyadic in the exponent of the wave function increases very considerably the labor required for the calculation.

Finally, we might expect, also by analogy with the results of the scalar variational method, that if the free-space Green's function $\mathbf{G}(\mathbf{r},\mathbf{r}')$ is chosen as the trial solution for the point-source wave function

[17] Note that if either x or y, or both, are zero, $\mathbf{Y}$ is directly summable without transformation.

[18] S. Altshuler, M. M. Moe, and P. Molmud, Space Technology Laboratories, Rept. GM-TR-0165-00397, 1958 (unpublished).

[19] The axially symmetric problem has been presented here mainly for the purpose of illustrating the application of the exponential variational principle. Further discussion of the physical problem, and the validity of the approximate solutions may be found in reference 18.

$\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$ in the stationary form, Eq. (10), the resulting equation should represent, in the limit of large k, the free-wave Green's function corrected for phase change along straight-line trajectories connecting the points $\mathbf{p}$ and $\mathbf{r}$; in addition to phase corrections, the improved wave function will also contain polarization corrections because of the dyadic expression in the exponent of Eq. (10). The inverse function $\mathbf{\Gamma}^{-1}(\mathbf{r},\mathbf{r}')$, appearing in Eq. (10), is, of course, the matrix inverse of $\mathbf{G}(\mathbf{r},\mathbf{r}')$:

$$\mathbf{G}^{-1}(\mathbf{r},\mathbf{r}')=\left(\frac{e^{ikR}}{-4\pi R^3k^2}\right)^{-1}\frac{1}{A}\left[\mathbf{I}-\frac{\mathbf{RR}B}{2(1-ikR)}\right],$$

where

$$\mathbf{R}=\mathbf{r}-\mathbf{r}',$$

$$A=k^2R^2+ikR-1,$$

$$B=-k^2-\frac{3ik}{R}+\frac{3}{R^2}.$$

VI. SUMMARY

A selection of variational principles for solutions of the vector wave equation—describing the scattering of an electromagnetic wave from an isotropic, nonhomogeneous potential—have been presented; these principles, of course, by no means exhaust the possible stationary expressions. Indeed, using the stationarity property of the forms presented in this paper, one may easily construct an infinite class of variational principles. However, detailed analysis of the stationary character of the variational principles is required before intuition and guesswork can be replaced by a positive criterion for the best choice of variational principle appropriate to any given problem.

Although the scattering medium has been assumed to be isotropic, this restriction is not necessary and may be relaxed somewhat. It is readily established that all the results are equally valid if the potential $U(\mathbf{r})$ is a *symmetric* matrix, provided only that care is exercised in maintaining the proper order of factors. Furthermore, the variational principles based on the integral equation, for example, Eq. (22), are valid for a completely general nonsymmetric potential, although those based on the differential operator are not.

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APPENDIX I

In this Appendix, we wish to prove the reciprocity relation for the Green's function $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$. Although the scattering potential has been assumed to be isotropic, all the results of the text apply equally well to an anisotropic, but symmetric, potential. Thus, the symmetry relation $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')=\mathbf{\Gamma}^T(\mathbf{r}',\mathbf{r})$ will be established for the more general case.

If, from the equation

$$\nabla\times\nabla\times\mathbf{\Gamma}(\mathbf{r},\mathbf{r}_0)-k^2\mathbf{\Gamma}(\mathbf{r},\mathbf{r}_0)+\mathbf{U}(\mathbf{r})\mathbf{\Gamma}(\mathbf{r},\mathbf{r}_0)=-\mathbf{I}\delta(\mathbf{r}-\mathbf{r}_0),$$

pre-multiplied by $\mathbf{\Gamma}^T(\mathbf{r},\mathbf{r}_1)$, is subtracted the transpose of the equation

$$\nabla\times\nabla\times\mathbf{\Gamma}(\mathbf{r},\mathbf{r}_1)-k^2\mathbf{\Gamma}(\mathbf{r},\mathbf{r}_1)+\mathbf{U}(\mathbf{r})\mathbf{\Gamma}(\mathbf{r},\mathbf{r}_1)=-\mathbf{I}\delta(\mathbf{r}-\mathbf{r}_1),$$

post-multiplied by $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}_0)$, and the result integrated over all $\mathbf{r}$ space, there results

$$\int\{\mathbf{\Gamma}^T(\mathbf{r},\mathbf{r}_1)\nabla\times\nabla\times\mathbf{\Gamma}(\mathbf{r},\mathbf{r}_0)+\mathbf{\Gamma}^T(\mathbf{r},\mathbf{r}_1)\mathbf{U}(\mathbf{r})\mathbf{\Gamma}(\mathbf{r},\mathbf{r}_0)$$
$$-[\nabla\times\nabla\times\mathbf{\Gamma}(\mathbf{r},\mathbf{r}_1)]^T\mathbf{\Gamma}(\mathbf{r},\mathbf{r}_0)-\mathbf{\Gamma}^T(\mathbf{r},\mathbf{r}_1)\mathbf{U}^T(\mathbf{r})\mathbf{\Gamma}(\mathbf{r},\mathbf{r}_0)\}d\mathbf{r}$$
$$=\int[\mathbf{\Gamma}(\mathbf{r},\mathbf{r}_0)\delta(\mathbf{r}-\mathbf{r}_1)-\mathbf{\Gamma}^T(\mathbf{r},\mathbf{r}_1)\delta(\mathbf{r}-\mathbf{r}_0)]d\mathbf{r}.$$

Using the assumption of a symmetric potential and the identity, Eq. (7), the left-hand side may be simplified and converted to a surface integral; thus, we have for the i, k component

$$\int\{\Gamma_{il}^T(\mathbf{r},\mathbf{r}_1)\partial_j\Gamma_{jk}(\mathbf{r},\mathbf{r}_0)-\Gamma_{ij}^T(\mathbf{r},\mathbf{r}_1)\partial_l\Gamma_{jk}(\mathbf{r},\mathbf{r}_0)$$
$$+\partial_l\Gamma_{ij}^T(\mathbf{r},\mathbf{r}_1)\Gamma_{jk}(\mathbf{r},\mathbf{r}_0)-\partial_j\Gamma_{ij}^T(\mathbf{r},\mathbf{r}_1)\Gamma_{lk}(\mathbf{r},\mathbf{r}_0)\}dS_l$$
$$=\Gamma_{ik}(\mathbf{r}_1,\mathbf{r}_0)-\Gamma_{ik}^T(\mathbf{r}_0,\mathbf{r}_1).$$

If the surface of integration is chosen to be a very large sphere, then on the surface $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}_0)$ and $\mathbf{\Gamma}^T(\mathbf{r},\mathbf{r}_1)$ must both satisfy the boundary condition that they be transverse; i.e., from the asymptotic forms of the integral equations, $\hat{r}\cdot\mathbf{\Gamma}\sim0$, $\mathbf{\Gamma}^T\cdot\hat{r}\sim0$. Thus, $\Gamma^T{}_{il}dS_l$ and $\Gamma_{lk}dS_l$ vanish asymptotically and the first and fourth terms of the integrand are, therefore, zero on the surface of integration. Again, using the asymptotic forms

$$\mathbf{\Gamma}(\mathbf{r},\mathbf{r}_0)\sim\frac{e^{ikr}}{-4\pi r}\mathbf{A}^T(-\mathbf{k}|\mathbf{r}_0)$$

$$\mathbf{\Gamma}^T(\mathbf{r},\mathbf{r}_1)\sim\frac{e^{ikr}}{-4\pi r}\mathbf{A}(-\mathbf{k}|\mathbf{r}_1),\quad \hat{k}=\hat{r}$$

and the relation $dS_l\partial_l=dS\partial/\partial r$, the second and third terms of the integrand become

$$dS_l\partial_l\Gamma_{ij}^T(\mathbf{r},\mathbf{r}_1)\Gamma_{jk}(\mathbf{r},\mathbf{r}_0)-\Gamma_{ij}^T(\mathbf{r},\mathbf{r}_1)dS_l\partial_l\Gamma_{jk}(\mathbf{r},\mathbf{r}_0)$$
$$=dS\left[\frac{\partial}{\partial r}\frac{e^{ikr}}{-4\pi r}\right]\frac{e^{ikr}}{-4\pi r}[A_{ij}(-\mathbf{k}|\mathbf{r}_1)A_{jk}^T(-\mathbf{k}|\mathbf{r}_0)-A_{ij}(-\mathbf{k}|\mathbf{r}_1)A_{jk}^T(-\mathbf{k}|\mathbf{r}_0)]$$
$$=0.$$

Therefore, $\mathbf{\Gamma}(\mathbf{r}_1,\mathbf{r}_0)=\mathbf{\Gamma}^T(\mathbf{r}_0,\mathbf{r}_1)$.

The analogous "reciprocity" relation may be established in the same way for a general nonsymmetric tensor potential, with the result

$$\mathbf{\Gamma}(\mathbf{r}',\mathbf{r})=\mathbf{\Gamma}^T(\mathbf{r},\mathbf{r}')+2\int \mathbf{\Gamma}^T(\mathbf{r}'',\mathbf{r}')\mathbf{U}_a(\mathbf{r}'')\mathbf{\Gamma}(\mathbf{r}'',\mathbf{r})d\mathbf{r}'',$$

where $\mathbf{U}_a(\mathbf{r})$ is the purely antisymmetric part of the potential $\mathbf{U}(\mathbf{r})$.

APPENDIX II

We wish to prove the stationarity of the variational principle $\mathbf{Y}_1$[Eq. (14)]. To first order in the variations, we have

$$\delta\mathbf{Y}_1=\delta\mathbf{A}(\mathbf{k}_0|\mathbf{r})+\int \delta\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')[\nabla'\times\nabla'\times-k^2+U(\mathbf{r}')]\cdot\mathbf{A}'d\mathbf{r}'+\int \mathbf{\Gamma}(\mathbf{r},\mathbf{r}')[\nabla'\times\nabla'\times-k^2+U(\mathbf{r}')]\delta\mathbf{A}'d\mathbf{r}',$$

where $\mathbf{A}'=\mathbf{A}(\mathbf{k}_0|\mathbf{r}')$. Use of the differential equations for $\mathbf{A}'$ and $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$, and Eq. (7), then leads to

$$\begin{aligned}\delta\mathbf{Y}_1&=\delta\mathbf{A}(\mathbf{k}_0|\mathbf{r})+\int\{[\nabla'\times\nabla'\times\mathbf{\Gamma}(\mathbf{r}',\mathbf{r})]^T\delta\mathbf{A}'\\&\quad-[(k^2-U(\mathbf{r}'))\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')]\delta\mathbf{A}'\}d\mathbf{r}'+\text{surface integral}\\&=\delta\mathbf{A}(\mathbf{k}_0|\mathbf{r})-\delta\mathbf{A}(\mathbf{k}_0|\mathbf{r})+\text{surface integral},\end{aligned}$$

where the i, k element of the surface integral is

$$J_{ik}=\int dS_l'\{\Gamma_{il}(\mathbf{r},\mathbf{r}')[\partial_j\delta A_{jk}']-\Gamma_{ij}(\mathbf{r},\mathbf{r}')[\partial_l\delta A_{jk}']+[\partial_l\Gamma_{ij}(\mathbf{r},\mathbf{r}')]\delta A_{jk}'-[\partial_j\Gamma_{ij}(\mathbf{r},\mathbf{r}')]\delta A_{lk}'\}.$$

Choosing the surface of integration to be the sphere at infinity and using the asymptotic form of $\Gamma(\mathbf{r},\mathbf{r}')$ for large r',

$$\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')\sim\frac{e^{ikr'}}{-4\pi r'}\mathbf{A}(-\mathbf{k}'|\mathbf{r}),$$

results in the following expression for J_{ik}:

$$\begin{aligned}J_{ik}=\int dS_l'\bigg\{&\frac{e^{ikr'}}{-4\pi r'}A_{il}(-\mathbf{k}'|\mathbf{r})\partial_j\delta A_{jk}'\\&-\frac{e^{ikr'}}{-4\pi r'}A_{ij}(-\mathbf{k}'|\mathbf{r})\partial_l\delta A_{jk}'\\&+\partial_l\bigg[\frac{e^{ikr'}}{-4\pi r'}A_{ij}(-\mathbf{k}'|\mathbf{r})\bigg]\delta A_{jk}'\\&-\partial_j\bigg[\frac{e^{ikr'}}{-4\pi r'}A_{ij}(-\mathbf{k}'|\mathbf{r})\bigg]\delta A_{lk}'\bigg\}.\end{aligned}$$

The first term of the integrand is zero, since

$$A_{il}(-\mathbf{k}'|\mathbf{r})dS_l'=[\mathbf{A}(-\mathbf{k}'|\mathbf{r})\cdot\hat{k}']_i dS'$$

which vanishes, according to Eq. (9).

The quantity $dS_l'\delta A_{lk}'$, appearing in the last term of the integrand, is also zero, for

$$dS_l'\delta A_{lk}'=dS_l'\delta A_{lk}(\mathbf{k}_0|\mathbf{r}')\sim dS_l'\frac{e^{ikr'}}{r'}\delta F_{lk}(\mathbf{k}_0|\mathbf{k}')$$

or

$$dS_l'\delta A_{lk}'\sim dS'\frac{e^{ikr'}}{r'}[\hat{k}'\cdot\delta\mathbf{F}(\mathbf{k}_0|\mathbf{k}')]_k;$$

but $\hat{k}'\cdot\mathbf{F}(\mathbf{k}_0|\mathbf{k}')=0$, and if the same transversality property is assumed for the trial functions, then

$$\hat{k}'\cdot\delta\mathbf{F}(\mathbf{k}_0|\mathbf{r}')=0.$$

Finally, using $dS_l'\partial_l=\partial/\partial r'$ and $\delta A_{jk}(\mathbf{k}_0|\mathbf{r}')\sim(e^{ikr'}/r')\times\delta F_{jk}(\mathbf{k}_0|\mathbf{k}')$, the second and third terms of the integrand are observed to cancel. Thus, $J_{ik}=0$, as was to be proved.